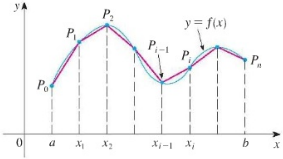
Arc Lenth

In this section, we derive a formula for the length of a curve $y = f(x)$ on an interval $[a, b]$. We will assume that f is continuous and differentiable on the interval $[a, b]$ and we will assume that its derivative f' is also continuous on the interval $[a, b]$. We use Riemann sums to approximate the length of the curve over the interval and then take the limit to get an integral.



We see from the picture that

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Letting $\Delta x = \frac{b-a}{n} = |x_{i-1} - x_i|$, we get

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \Delta x \sqrt{1 + \left[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right]^2}$$

Now by the mean value theorem from last semester, we have $\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = f'(x_i^*)$ for some x_i^* in the interval $[x_{i-1}, x_i]$. Therefore

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2} \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Arc Length

If f is continuous and differentiable on the interval $[a, b]$ and f' is also continuous on the interval $[a, b]$. We have a formula for the length of a curve $y = f(x)$ on an interval $[a, b]$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx \quad \text{or} \quad L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}dx$$

Example Find the arc length of the curve $y = \frac{2x^{3/2}}{3}$ from $(1, \frac{2}{3})$ to $(2, \frac{4\sqrt{2}}{3})$.

- $f(x) = \frac{2x^{3/2}}{3}$, $\quad f'(x) = \frac{3}{2} \cdot \frac{2}{3}x^{1/2} = \sqrt{x}$, $\quad [f'(x)]^2 = x$, $\quad a = 1$ and $b = 2$.
- $L = \int_a^b \sqrt{1 + [f'(x)]^2}dx = \int_1^2 \sqrt{1 + x}\, dx$
- $= \int_2^3 \sqrt{u}\, du$, where $u = 1 + x$, $u(1) = 2$, $\quad u(2) = 3$.
- $= \left.\frac{u^{3/2}}{3/2}\right|_2^3 = \frac{2}{3}[3\sqrt{3} - 2\sqrt{2}]$.

Arc Length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad \text{or} \quad L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx$$

Example Find the arc length of the curve $y = \frac{e^x + e^{-x}}{2}, \quad 0 \le x \le 2.$

- $f(x) = \frac{e^x + e^{-x}}{2}, \quad f'(x) = \frac{e^x - e^{-x}}{2},$
  $[f'(x)]^2 = \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} = \frac{e^{2x} - 2 + e^{-2x}}{4}, \quad a = 0$ and $b = 2.$
- $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_0^2 \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}}\, dx = \int_0^2 \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}}\, dx$
- (By an application of Pilkington's Law :) which says that if $AB = 1/4$, then $1 + (A - B)^2 = (A + B)^2$.)
- $= \int_0^2 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx = \int_0^2 \frac{e^x + e^{-x}}{2}\, dx.$
- $= \left.\frac{e^x - e^{-x}}{2}\right|_0^2 = \frac{e^2 - e^{-2}}{2}.$

Arc Length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad \text{or} \quad L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx$$

Example Set up the integral which gives the arc length of the curve $y = e^x, \; 0 \leq x \leq 2$.

- $f(x) = e^x, \quad f'(x) = e^x, \quad [f'(x)]^2 = e^{2x}, \quad a = 0$ and $b = 2$.
- $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx = \int_0^2 \sqrt{1 + e^{2x}} \, dx$.

Arc Length, curves of form $x = g(y)$.

For a curve with equation $x = g(y)$, where $g(y)$ is continuous and has a continuous derivative on the interval $c \le y \le d$, we can derive a similar formula for the arc length of the curve between $y = c$ and $y = d$.

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\,dy \quad \text{or} \quad L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\,dy$$

Example Find the length of the curve $24xy = y^4 + 48$ from the point $(\frac{4}{3}, 2)$ to $(\frac{11}{4}, 4)$.

- Solving for x in terms of y, we get $x = \frac{y^4+48}{24y} = \frac{y^3}{24} + \frac{2}{y} = g(y)$.
- $g'(y) = \frac{3y^2}{24} - \frac{2}{y^2} = \frac{y^2}{8} - \frac{2}{y^2}$. $[g'(y)]^2 = \frac{y^4}{64} - \frac{4}{y^2} \cdot \frac{y^2}{8} + \frac{4}{y^4} = \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}$
- $L = \int_c^d \sqrt{1 + [f'(y)]^2}dy = \int_2^4 \sqrt{1 + \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}}\, dy$.
- $= \int_2^4 \sqrt{\frac{y^4}{64} + \frac{1}{2} + \frac{4}{y^4}}\, dy = \int_2^4 \sqrt{\left(\frac{y^2}{8} + \frac{2}{y^2}\right)^2}\, dy$.
- $= \int_2^4 \frac{y^2}{8} + \frac{2}{y^2}\, dy = \frac{y^3}{24} - \frac{2}{y}\Big|_2^4 = \frac{17}{6}$.

Arc Length, Simpson's rule

We cannot always find an antiderivative for the integrand to evaluate the arc length. However, we can use Simpson's rule to estimate the arc length.

Example Use Simpson's rule with $n = 10$ to estimate the length of the curve

$$x = y + \sqrt{y}, \quad 2 \leq y \leq 4$$

- $dx/dy = 1 + \frac{1}{2\sqrt{y}}$,
- $L = \int_2^4 \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy = \int_2^4 \sqrt{1 + \left[1 + \frac{1}{2\sqrt{y}}\right]^2}\, dy = \int_2^4 \sqrt{2 + \frac{1}{\sqrt{y}} + \frac{1}{4y}}\, dy$
- With $n = 10$, Simpson's rule gives us

$$L \approx S_{10} = \frac{\Delta y}{3}\left[g(2) + 4g(2.2) + 2g(2.4) + 4g(2.6) + 2g(2.8) + 4g(3) + 2g(3.2) + 4g(3.4) + 2g(3.6) + 4g(3.8) + g(4)\right]$$

where $g(y) = \sqrt{2 + \frac{1}{\sqrt{y}} + \frac{1}{4y}}$ and $\Delta y = \frac{4-2}{10}$. (see notes for details).

- We get $S_{10} \approx 3.269185$.

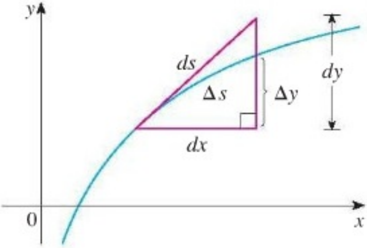
Arc Length Function

The distance along a curve with equation $y = f(x)$ from a fixed point $(a, f(a))$ is a function of x. It is called the **arc length function** and is given by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

From the fundamental theorem of calculus, we see that $s'(x) = \sqrt{1 + [f'(x)]^2}$. In the language of differentials, this translates to

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad (ds)^2 = (dx)^2 + (dy)^2$$



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Example Find the arc length function for the curve $y = \frac{2x^{3/2}}{3}$ taking $P_0(1, 3/2)$ as the starting point.

- We have $\frac{d \frac{2x^{3/2}}{3}}{dx} = \sqrt{x}$
- $$s(x) = \int_1^x \sqrt{1 + (\sqrt{t})^2} dt = \int_1^x \sqrt{1 + t} dt = \int_2^{1+x} \sqrt{u} du$$
  where $u = 1 + t$
- $$= \frac{u^{3/2}}{3/2}\Bigg|_2^{x+1} = 2(x+1)\sqrt{x+1}/3 - 4\sqrt{2}/3$$